Pergamon

Int. J. Solids Structures Vol. 32, No. 23, pp. 3515–3531, 1995

0020–7683/95 $9.50 + .00

0020–7683(95)00005–4

A PLANE STRAIN FORMULATION OF THE ELASTIC–PLASTIC CONSTITUTIVE LAW FOR HARDENING VON MISES MATERIALS

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(*Received* 20 *July* 1994; *in revised form* 22 *December* 1994)

Abstract—Elastic–plastic (path dependent but time-independent) materials, linear and isotropic in the elastic range and governed by the von Mises yield condition and the relevant associated flow rule, are considered. Both kinematic and isotropic hardening are included. Starting from previous, more general but partial, results and exploiting the purely deviatoric nature of plastic flow, a plane strain version of the constitutive law is formulated in terms of in-plane variables only. The actual plastic strains are replaced by equivalent in-plane measures and the effects of transverse yielding appear as a fictitious, directional kinematic hardening. On this basis, the plane strain elastic–plastic problem can be formulated in a way fully analogous to the equivalent plane stress one: as in the elastic case, differences are confined to the expressions of the constitutive parameters.

1. INTRODUCTION

The linear elastic isotropic plane strain problem can be formulated in terms of the in-plane variables only; the transverse normal stress σ_z does not vanish, but can be computed once the in-plane solution has been obtained. In the inelastic case, however, the constitutive law usually depends on σ_z in a complicated manner and its elimination is not equally easy. In computations, often the same constitutive relations as for the three-dimensional case are employed, thereby exploiting only partially the plane nature of the problem.

A plane strain formulation of the rate constitutive law for standard elastic–plastic materials was proposed by Corradi and Gioda (1979). The plane strain hypotheses and the elastic portion of the constitutive relation permitted the expression of σ_z as a function of the remaining stress and of the non-vanishing plastic strain components. An equivalent, in-plane yield function was defined on this basis, which did not depend explicitly on σ_z, and it was shown that the associated flow rule remained valid provided the actual plastic strains were replaced by equivalent in-plane measures, accounting for plasticity in the transverse direction. The effects of transverse yielding appeared as a fictitious hardening in the in-plane representation.

The formulation, however, was not completely "plane", in that the yield function was affected by the current value of the transverse plastic strain component p_z, which appeared as an additional material state variable. Since the state is supposed to be known at the instant at which the rate law applies, this is perfectly acceptable from a theoretical point of view. When performing computations, however, the presence of p_z is a source of difficulties and its elimination is desirable. This result is readily achieved if piecewise linear elastic–plastic relations are used (Cohn and Maier, 1979), and in this case the plane strain problem differs from the equivalent plane stress one only for the constitutive law to be applied in either case. Computations performed by Corradi and Gioda (1980) for axisymmetric cylinders demonstrated the effectiveness of the formulation.

Even if meaningful, this is however a rather restrictive case and in general the elimination of p_z is not equally easy. The problem arises because, due to the irreversible nature of plastic behaviour, the constitutive relations can only be written in terms of *rates*. The rates of p_z and of the in-plane plastic strain components are related by the normality law, but not the total values accumulated in the preceding plastic evolution. For a number of yield conditions of engineering importance, however, normality implies that plastic strain rates are purely deviatoric, a feature which is shared by total values as well and permits the elimination of all transverse components from the plane strain law.

In this paper, such a law is constructed for materials which are linear and isotropic in the elastic range and are governed by the von Mises yield condition, accounting for, possibly nonlinear, kinematic and isotropic hardening. For the sake of simplicity, thermal effects are not considered; however both thermal strains and dependence of the yield condition on temperature, in the form introduced by Prager (see Naghdi 1960), can be easily included. Their influence was discussed in Corradi and Gioda (1979) and no difference whatsoever arises in the present context.

Notation

A vectorial notation is mainly used. Column vectors and matrices are denoted by bold face symbols. A row vector is indicated as the transpose of a column one. The derivative of column vector $\mathbf{a}$ with respect to column vector $\mathbf{b}$ is defined so that the resulting matrix $\mathbf{M} = \partial\mathbf{a}/\partial\mathbf{b}$ has components $M_{ij} = \partial a_j/\partial b_i$. The derivative of a scalar with respect to a column (row) vector is therefore a column (row) vector. Moreover for any scalar function $f(\mathbf{a})$, with $\mathbf{a} = \mathbf{A}\mathbf{b}$, the chain rule of differentiation states

$$\frac{\partial f}{\partial \mathbf{b}} = \frac{\partial \mathbf{a}}{\partial \mathbf{b}}\frac{\partial f}{\partial \mathbf{a}} = \mathbf{A}^{\mathrm{T}}\frac{\partial f}{\partial \mathbf{a}}.$$

Superscripts T and -1 mean *transpose* and *inverse*, respectively. A superposed dot denotes the time derivative. When tensorial notation is used, summation on repeated indices is understood.

2. REVIEW OF ELASTIC–PLASTIC LAWS FOR STANDARD MATERIALS

Strains are supposed to be small enough to be treated as infinitesimal quantities. In the absence of temperature changes, their total values ε_{ij} can be decomposed as the sum of elastic and plastic contributions, the latter denoted by ε^p_{ij}. For an elastically linear and isotropic material, total strains can be written as

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \varepsilon^p_{ij} \tag{1}$$

where δ_{ij} is the unit (Kronecker) second order tensor. The plastic behaviour of the material is governed by the von Mises yield condition. The loading (or yield) function is

$$\phi(\sigma_{ij}, \varepsilon^p_{ij}, \kappa) = [\tfrac{3}{2}(s_{ij}-\chi_{ij})(s_{ij}-\chi_{ij})]^{1/2} - \sigma_0(\kappa) \tag{2}$$

where σ_0 is the instantaneous uniaxial yield limit, s_{ij} is the stress deviator tensor and

$$\chi_{ij}(\varepsilon^p_{ij}) = \frac{\partial V(\varepsilon^p_{ij})}{\partial \varepsilon^p_{ij}} \tag{3}$$

is the so-called *back-stress* tensor, also of deviatoric nature, accounting for kinematic hardening. Equation (3) expresses it as the gradient of a twice differentiable and convex function of the current value of total plastic strains, denoted by $V = V(\varepsilon^p_{ij})$. The simplest assumption is

$$V(\varepsilon^p_{ij}) = \tfrac{1}{2}c\varepsilon^p_{ij}\varepsilon^p_{ij} \qquad \chi_{ij} = c\varepsilon^p_{ij} \qquad (c > 0) \tag{4a,b}$$

and is referred to as linear kinematic hardening. What follows, however, is not limited to this case.

Isotropic hardening is also present in eqn (2), since the yield limit σ_0 is a (non-decreasing) function of the internal variable $\kappa = \int_0^t \dot{\kappa}(\tau)\,d\tau$, with $\dot{\kappa} > 0$ when plastic flow actually occurs. The two most widely used expressions, known as *strain-hardening* and *work-hardening*, respectively, are

$$\dot{\kappa} = (\tfrac{2}{3}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij})^{1/2} \qquad \dot{\kappa} = \sigma_{ij}\dot{\varepsilon}^p_{ij}. \tag{5a,b}$$

The instantaneous elastic range is defined by the inequality $\phi \leqslant 0$. When $\phi = 0$, plastic strain rates may develop and are governed by the *associated* flow rule and Prager's consistency condition. They can be expressed as follows (Maier, 1969)

$$\dot{\varepsilon}^p_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}}\dot{\lambda} \qquad \dot{\phi} = \frac{\partial \phi}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - h\dot{\lambda} \tag{6a,b}$$

with

$$\dot{\lambda} = 0 \qquad \text{if} \quad \phi < 0 \tag{7a}$$

$$\dot{\lambda} \geqslant 0,\ \dot{\phi} \leqslant 0,\ \dot{\phi}\dot{\lambda} = 0 \quad \text{if} \quad \phi = 0. \tag{7b}$$

Equation (6a) is often referred to as the *normality law*. In eqn (6b), h is the hardening parameter, non-negative in the present context. Comparison between eqn (6b) and the expression of the time derivative of ϕ establishes

$$-h\dot{\lambda} = \frac{\partial \phi}{\partial \varepsilon^p_{ij}}\dot{\varepsilon}^p_{ij} + \frac{\partial \phi}{\partial \kappa}\dot{\kappa} = \frac{\partial \chi_{hl}}{\partial \varepsilon^p_{ij}}\frac{\partial \phi}{\partial \chi_{hl}}\dot{\varepsilon}^p_{ij} - \frac{\partial \sigma_0}{\partial \kappa}\dot{\kappa}.$$

Since

$$\frac{\partial \phi}{\partial \chi_{hl}} = -\frac{\partial \phi}{\partial s_{hl}} = -\frac{\partial \phi}{\partial \sigma_{hl}}$$

and because of eqns (3) and (6a), it is

$$h = \frac{\partial^2 V}{\partial \varepsilon^p_{ij}\,\partial \varepsilon^p_{hl}}\frac{\partial \phi}{\partial \sigma_{ij}}\frac{\partial \phi}{\partial \sigma_{hl}} + \frac{\partial \sigma_0}{\partial \kappa}h_l \tag{8a}$$

where $\dot{\kappa}$ was expressed as $\dot{\kappa} = h_l\dot{\lambda}$. In particular, from eqns (5) one obtains

$$h_l = \left(\frac{2}{3}\frac{\partial \phi}{\partial \sigma_{ij}}\frac{\partial \phi}{\partial \sigma_{ij}}\right)^{1/2} \quad \text{or} \quad h_l = \sigma_{ij}\frac{\partial \phi}{\partial \sigma_{ij}}. \tag{8b,c}$$

3. PLANE STRAIN FORMULATION

The above relations will now be specialized for a body in a state of plane strain normal to the $z = x_3$ axis. The following three-component vectors are introduced

$$\boldsymbol{\varepsilon} = [\varepsilon_x = \varepsilon_{11}, \varepsilon_y = \varepsilon_{22}, \gamma_{xy} = 2\varepsilon_{12}]^T \tag{9a}$$

$$\mathbf{p} = [p_x = \varepsilon^p_{11}, p_y = \varepsilon^p_{22}, p_{xy} = 2\varepsilon^p_{12}]^T \tag{9b}$$

$$\boldsymbol{\sigma} = [\sigma_x = \sigma_{11}, \sigma_y = \sigma_{22}, \sigma_{xy} = \sigma_{12}]^T \tag{9c}$$

$$\boldsymbol{\chi} = [\chi_x = \chi_{11}, \chi_y = \chi_{22}, \chi_{xy} = \chi_{12}]^T \tag{9d}$$

The plane strain hypothesis states that $\varepsilon_z = \varepsilon_{33}$, $\gamma_{zx} = 2\varepsilon_{31}$ and $\gamma_{zy} = 2\varepsilon_{32}$ are equal to zero.

The constitutive law under consideration also entails $(\cdot)_{31} = (\cdot)_{32} = 0$ for all variables. However, the transverse components

$$p_z = \varepsilon^p_{33}, \quad \sigma_z = \sigma_{33}, \quad \chi_z = \chi_{33} \tag{9e}$$

do not vanish. Note that, as eqns (9a,b) show, for shearing strains the engineering definition, twice the corresponding tensorial component, is adopted. It follows, in particular

$$\varepsilon^p_{ij}\varepsilon^p_{ij} = \mathbf{p}^{\mathrm{T}}\mathbf{C}\mathbf{p} + p_z^2 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \tag{10a}$$

$$\sigma_{ij}\dot{\varepsilon}^p_{ij} = \boldsymbol{\sigma}^{\mathrm{T}}\dot{\mathbf{p}} + \sigma_z\dot{p}_z. \tag{10b}$$

Function $V = V(\mathbf{p}, p_z)$ is defined so that eqn (3) reads

$$\boldsymbol{\chi} = \frac{\partial V}{\partial \mathbf{p}}, \quad \chi_z = \frac{\partial V}{\partial p_z} \quad \text{with} \quad V = V(\mathbf{p}, p_z). \tag{11}$$

The elastic portion of the constitutive law can now be split in the pair of relations

$$\boldsymbol{\sigma} = \mathbf{D}[\boldsymbol{\varepsilon} - (\mathbf{p} + \nu\boldsymbol{\mu}p_z)] \qquad \sigma_z = \nu\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\sigma} - Ep_z \tag{12a,b}$$

where

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}, \qquad \boldsymbol{\mu} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{13a,b}$$

The vector

$$\mathbf{P} = \mathbf{p} + \nu\boldsymbol{\mu}p_z = [p_x + \nu p_z, p_y + \nu p_z, p_{xy}]^{\mathrm{T}} \tag{14}$$

defines the equivalent in-plane plastic strains, accounting for the transverse component contribution through the elastic Poisson ratio ν. Then, eqn (12a) becomes

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \mathbf{P}). \tag{15}$$

Equation (12b) permits the elimination of σ_z from the constitutive law. As was shown in Corradi and Gioda (1979), any loading function of the form $\phi = \phi(\boldsymbol{\sigma}, \sigma_z, \mathbf{p}, p_z, \kappa)$ can be transformed in the equivalent in-plane expression

$$\Phi(\boldsymbol{\sigma}, \mathbf{P}, p_z, \kappa) = \phi(\boldsymbol{\sigma}, \boldsymbol{\sigma}_z = \nu\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\sigma} - Ep_z, \mathbf{p} = \mathbf{P} - \nu\boldsymbol{\mu}p_z, p_z, \kappa). \tag{16}$$

Then, the rate law is expressed by the following relations, formally identical to eqns (6) and (7)

$$\dot{\mathbf{P}} = \mathbf{N}\dot{\lambda} \qquad \dot{\Phi} = \mathbf{N}^{\mathrm{T}}\dot{\boldsymbol{\sigma}} - H\dot{\lambda} \tag{17a,b}$$

$$\dot{\lambda} = 0 \qquad \text{if} \quad \Phi < 0 \tag{18a}$$

$$\dot{\lambda} \geqslant 0, \dot{\Phi} \leqslant 0, \dot{\Phi}\dot{\lambda} = 0 \quad \text{if} \quad \Phi = 0 \tag{18b}$$

with

$$\mathbf{N} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \frac{\partial \phi}{\partial \boldsymbol{\sigma}} + v\boldsymbol{\mu}\frac{\partial \phi}{\partial \sigma_z}, \qquad H = h + E\left(\frac{\partial \phi}{\partial \sigma_z}\right)^2. \tag{19a,b}$$

The value of the internal variable κ keeps unchanged, even if in the relevant expression of $\dot{\kappa}$, $\mathbf{p}$ is replaced by $\mathbf{P}$ and σ_z, if present, can be eliminated.

4. PLANE STRAIN FORM OF THE VON MISES PLASTICITY CONDITION

The plane strain loading function eqn (16) contains p_z as an additional material state variable. For some plasticity laws of engineering interest, however, no volume change is associated to plastic flow and this feature can be exploited to further simplify the plane strain law.

The von Mises condition belongs to this category and the procedure is now illustrated with reference to it. If written in terms of stress rather than stress deviator components, the expression (2) of the yield function reads, in the plane strain case

$$\phi = \alpha(\boldsymbol{\sigma}, \sigma_z, \boldsymbol{\chi}, \chi_z) - \sigma_0(\kappa) \leqslant 0 \tag{20a}$$

where

$$\alpha^2 = (\sigma_x - \chi_x)^2 - (\sigma_x - \chi_x)(\sigma_y - \chi_y) + (\sigma_y - \chi_y)^2 + 3(\tau_{xy} - \chi_{xy})^2 + (\sigma_z - \chi_z)^2 - (\sigma_x - \chi_x)(\sigma_z - \chi_z) - (\sigma_y - \chi_y)(\sigma_z - \chi_z) \tag{20b}$$

and back-stresses are given as functions of non-vanishing plastic strain components by eqn (11). Let the following matrix be introduced

$$\mathbf{m} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{21}$$

and note that from the expression (13b) of vector $\boldsymbol{\mu}$ it follows $\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\mu} = 2$ and $\mathbf{m}\boldsymbol{\mu} = \boldsymbol{\mu}$. Then, eqn (20b) can be compactly written as

$$\alpha^2 = \tfrac{1}{2}(\boldsymbol{\sigma}_* - \boldsymbol{\chi}_*)^{\mathrm{T}}\mathbf{m}(\boldsymbol{\sigma}_* - \boldsymbol{\chi}_*) \tag{22}$$

with

$$\boldsymbol{\sigma}_* = \boldsymbol{\sigma} - \boldsymbol{\mu}\sigma_z \quad \boldsymbol{\chi}_* = \boldsymbol{\chi} - \boldsymbol{\mu}\chi_z. \tag{23a,b}$$

The plastic incompressibility condition reads

$$p_z = -(p_x + p_y) = -\boldsymbol{\mu}^{\mathrm{T}}\mathbf{p}. \tag{24}$$

By introducing the above relation into eqn (14), one obtains

$$\mathbf{P} = \mathbf{A}\mathbf{p} \qquad \mathbf{A} = \mathbf{I} - v\boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} = \begin{bmatrix} 1-v & -v & 0 \\ -v & 1-v & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{25a,b}$$

Since the material is elastically compressible ($v < 0.5$), matrix $\mathbf{A}$ is non-singular and eqn (25a) can be inverted to give

$$\mathbf{p} = \mathbf{A}^{-1}\mathbf{P} \qquad \mathbf{A}^{-1} = \frac{1}{1-2\nu}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix}. \tag{26a,b}$$

It can be easily verified by direct substitution that the following equalities hold

$$\mathbf{A}\boldsymbol{\mu} = (1-2\nu)\boldsymbol{\mu} \quad \mathbf{A}^{-1}\boldsymbol{\mu} = \frac{1}{1-2\nu}\boldsymbol{\mu}. \tag{27a,b}$$

By introducing eqn (26a) into eqn (24), account taken of eqn (27b), one obtains

$$p_z = -\boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{P} = -\frac{1}{1-2\nu}\boldsymbol{\mu}^{\mathrm{T}}\mathbf{P}. \tag{28}$$

The expression (12b) of the transverse stress component then becomes

$$\sigma_z = \nu\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\sigma} + \frac{E}{1-2\nu}\boldsymbol{\mu}^{\mathrm{T}}\mathbf{P}. \tag{29}$$

This value can be substituted into eqn (23a), to get

$$\boldsymbol{\sigma}_* = \mathbf{A}\boldsymbol{\sigma} - \frac{E}{1-2\nu}\boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}}\mathbf{P} = \mathbf{A}\left[\boldsymbol{\sigma} - \frac{E}{(1-2\nu)^2}\boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}}\mathbf{P}\right].$$

By introducing the *fictitious back-stress* vector

$$\boldsymbol{\Pi} = E\boldsymbol{\Gamma}\mathbf{P}, \qquad \boldsymbol{\Gamma} = \frac{1}{(1-2\nu)^2}\boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} = \frac{1}{(1-2\nu)^2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{30a,b}$$

the above relation becomes

$$\boldsymbol{\sigma}_* = \mathbf{A}[\boldsymbol{\sigma} - \boldsymbol{\Pi}]. \tag{31}$$

Now let the function $V(\mathbf{p}, p_z)$ be expressed in terms of $\mathbf{P}$ by means of eqns (26a), (28) and an equivalent, in-plane back-stress vector $\boldsymbol{\beta} = \partial V/\partial\mathbf{P}$ be introduced. Account taken of symmetry of $\mathbf{A}$, the chain rule of differentiation establishes

$$\boldsymbol{\beta} = \frac{\partial V}{\partial \mathbf{P}} = \frac{\partial \mathbf{p}}{\partial \mathbf{P}}\frac{\partial V}{\partial \mathbf{p}} + \frac{\partial p_z}{\partial \mathbf{P}}\frac{\partial V}{\partial p_z} = \mathbf{A}^{-1}(\boldsymbol{\chi} - \boldsymbol{\mu}\chi_z). \tag{32}$$

Hence

$$\boldsymbol{\chi}_* = \mathbf{A}\boldsymbol{\beta}. \tag{33}$$

If eqns (31) and (33) are introduced in eqn (22), one realizes that eqns (20) are replaced by the expression

$$\Phi(\boldsymbol{\sigma}, \mathbf{P}, \kappa) = [\tfrac{1}{2}(\boldsymbol{\sigma} - \mathbf{X})^{\mathrm{T}}\mathbf{M}(\boldsymbol{\sigma} - \mathbf{X})]^{1/2} - \sigma_0(\kappa) \tag{34a}$$

where

$$\mathbf{M} = \mathbf{AmA} = \begin{bmatrix} 2-2\nu+2\nu^2 & -1-2\nu+2\nu^2 & 0 \\ -1-2\nu+2\nu^2 & 2-2\nu+2\nu^2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{34b}$$

and

$$\mathbf{X} = \mathbf{\Pi}+\boldsymbol{\beta} = E\mathbf{\Gamma P}+\frac{\partial V}{\partial \mathbf{P}} = \mathbf{X}(\mathbf{P}). \tag{34c}$$

Equations (34) express the von Mises loading function in terms of in-plane variables only, namely the in-plane stress components collected in vector $\boldsymbol{\sigma}$ and the equivalent in-plane plastic strains $\mathbf{P}$ (isotropic hardening is always governed by the additional internal variable κ). $\mathbf{X}$ is the total, equivalent in-plane back-stress vector composed, as eqn (34c) shows, of two contributions: $\boldsymbol{\beta} = \partial V/\partial \mathbf{P}$ incorporates the actual material kinematic hardening, while $\mathbf{\Pi}$ accounts for the effects of yielding in the transverse direction, which appear as an additional kinematic hardening in the plane; since matrix $\mathbf{\Gamma}$ has rank one, this term makes the instantaneous elastic domain move in a definite direction: eqns (30) in fact show that it is always $\pi_x = \pi_y$ and $\pi_{xy} = 0$.

As an example, the in-plane back-stress vectors are computed for linear kinematic hardening. Because of eqns (4) and (10a), the plane strain expression of function V reads

$$V(\mathbf{p}, p_z) = \tfrac{1}{2}c(\mathbf{p}^{\mathrm{T}}\mathbf{C}\mathbf{p}+p_z^2).$$

By introducing eqns (26a) and (28), one obtains

$$V(\mathbf{P}) = \tfrac{1}{2}c\mathbf{P}^{\mathrm{T}}\mathbf{\Theta P} \qquad \mathbf{\Theta} = \mathbf{A}^{-1}\mathbf{CA}^{-1}+\mathbf{\Gamma} = 3\mathbf{M}^{-1}. \tag{35a,b}$$

Hence

$$\boldsymbol{\beta} = 3c\mathbf{M}^{-1}\mathbf{P} \qquad \mathbf{X} = [E\mathbf{\Gamma}+3c\mathbf{M}^{-1}]\mathbf{P}. \tag{36a,b}$$

5. THE RATE LAW

The rate relations, eqns (17) and (18) associated to the plane strain von Mises loading function, eqns (34), are now established, i.e. the expressions for the in-plane outward normal vector $\mathbf{N}$ and of the equivalent in-plane hardening parameter H are derived. Since plastic flow can actually occur only when $\Phi = 0$, the equality

$$\alpha = [\tfrac{1}{2}(\boldsymbol{\sigma}-\mathbf{X})^{\mathrm{T}}\mathbf{M}(\boldsymbol{\sigma}-\mathbf{X})]^{1/2} = \sigma_0(\kappa) \tag{37}$$

is assumed to hold in the sequel.

The normal vector $\mathbf{N}$ is straightforwardly obtained by writing

$$\mathbf{N} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \frac{1}{2\sigma_0}\mathbf{M}(\boldsymbol{\sigma}-\mathbf{X}). \tag{38}$$

To express H, consider first that, by comparing eqn (17b) to the time derivative of Φ, namely

$$\dot{\Phi} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}^{\mathrm{T}}}\dot{\boldsymbol{\sigma}}+\frac{\partial \Phi}{\partial \mathbf{P}^{\mathrm{T}}}\dot{\mathbf{P}}+\frac{\partial \Phi}{\partial \kappa}\dot{\kappa}$$

one infers

$$H\dot{\lambda} = -\frac{\partial \Phi}{\partial \mathbf{P}^{\mathrm{T}}}\dot{\mathbf{P}} - \frac{\partial \Phi}{\partial \kappa}\dot{\kappa}. \tag{39}$$

The chain rule of differentiation and eqn (3a) state

$$\frac{\partial \Phi}{\partial \mathbf{P}} = \frac{\partial \mathbf{X}}{\partial \mathbf{P}}\frac{\partial \Phi}{\partial \mathbf{X}} = -\frac{\partial \mathbf{X}}{\partial \mathbf{P}}\mathbf{N}.$$

From eqn (34c) it also follows

$$\frac{\partial \mathbf{X}}{\partial \mathbf{P}} = \frac{\partial \mathbf{\Pi}}{\partial \mathbf{P}} + \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{P}} = E\Gamma + \frac{\partial}{\partial \mathbf{P}}\frac{\partial V}{\partial \mathbf{P}}.$$

By introducing the symmetric matrix

$$\mathbf{K} = \frac{\partial^2 V}{\partial \mathbf{P}^2} \quad \left(\mathrm{K}_{ij} = \frac{\partial^2 V}{\partial P_i \partial P_j}\right) \tag{40}$$

and by substituting eqn (17a) for **P**, one obtains

$$-\frac{\partial \Phi}{\partial \mathbf{P}^{\mathrm{T}}}\dot{\mathbf{P}} = \mathbf{N}^{\mathrm{T}}(E\Gamma + \mathbf{K})\mathbf{N}\dot{\lambda}. \tag{41a}$$

On the other hand, since κ affects the loading function only through the instantaneous yield limit σ_0, the last term in eqn (39) reads

$$-\frac{\partial \Phi}{\partial \kappa}\dot{\kappa} = \frac{\partial \sigma_0}{\partial \kappa}h_I\dot{\lambda} \tag{41b}$$

where, as done when writing eqns (8), $\dot{\kappa}$ was expressed in the form

$$\dot{\kappa} = h_I\dot{\lambda}. \tag{42}$$

By comparing eqns (41) with eqn (39), one realizes that the hardening parameter H can be expressed as the sum of three contributions, namely

$$H = H_E + H_K + H_I \tag{43}$$

where

$$H_E = E\mathbf{N}^{\mathrm{T}}\Gamma\mathbf{N} \qquad H_K = \mathbf{N}^{\mathrm{T}}\mathbf{K}\mathbf{N} \qquad H_I = \frac{\partial \sigma_0}{\partial \kappa}h_I. \tag{44a–c}$$

The first governs the fictitious plane strain hardening. By substituting eqn (38) for **N**, one obtains after some straightforward algebraic manipulations

$$H_E = \frac{1}{4}\frac{E(1-2\nu)^2}{\sigma_0^2(\kappa)}[(\sigma_x - X_x) + (\sigma_y - X_y)]^2. \tag{45}$$

If the material is perfectly plastic (σ_0 = constant, $\mathbf{X} = \mathbf{\Pi}$) only this contribution is present.

The second contribution H_K, defined by eqn (44b), accounts for material kinematic hardening. If this is linear, from eqn (35) follows $\mathbf{K} = 3c\,\mathbf{M}^{-1}$. Then, by substituting eqn (38) for **N** and remembering that when plastic flow develops eqn (37) holds, one obtains

$$H_K = \tfrac{3}{2}c. \tag{46}$$

Finally, H_I incorporates the influence of previous plastic strain history on the instantaneous yield limit (isotropic hardening). Its expression depends on the nature of the function $\sigma_0 = \sigma_0(\kappa)$ and on the definition of the internal variable κ itself. In the strain-hardening case, eqns (5a) and (10a) establish for $\dot{\kappa}$ the expression

$$\dot{\kappa} = h_I^{\text{sh}}\dot{\lambda} = [\tfrac{2}{3}(\dot{\mathbf{p}}^{\text{T}}\mathbf{C}\dot{\mathbf{p}}+\dot{p}_z^2)]^{1/2}$$

and, by proceeding as in the previous case, one obtains

$$h_I^{\text{sh}} = 1. \tag{47a}$$

On the other hand, for work-hardening eqns (5b) and (10b) yield

$$\dot{\kappa} = h_I^{\text{wh}}\dot{\lambda} = \boldsymbol{\sigma}^{\text{T}}\dot{\mathbf{p}}+\sigma_z\dot{p}_z.$$

By using eqns (26), (28), (29) and (38), the following expression is arrived at

$$h_I^{\text{wh}} = \frac{1}{2\sigma_0}(\boldsymbol{\sigma}-\mathbf{X})^{\text{T}}\mathbf{M}(\boldsymbol{\sigma}-\boldsymbol{\Pi}).$$

By substituting $\mathbf{X}-\boldsymbol{\beta}$ for $\boldsymbol{\Pi}$ and remembering eqn (37), one obtains the alternative expression

$$h_I^{\text{wh}} = \sigma_0(\kappa)+\frac{1}{2\sigma_0}\boldsymbol{\beta}^{\text{T}}\mathbf{M}(\boldsymbol{\sigma}-\mathbf{X}). \tag{47b}$$

If material hardening is purely isotropic, $\beta = 0$ and the two expressions (47) for h_I become identical, provided that the relevant definitions for $\dot{\kappa}$ are given the same dimension for multiplying the first, or dividing the second, by $\sigma_0(\kappa)$. It is in fact well known that in this case the same behaviour can be represented by either definition. When kinematic and isotropic hardening are combined differences arise. Note, in particular, that h_I^{wh} depends explicitly on the equivalent in-plane back-stress vector $\boldsymbol{\beta}$. Note also that in eqn (43) the sum $h = H_K+H_I$ is the original, material hardening coefficient, which is now expressed in terms of in-plane variables, but keeps its value unchanged.

6. ILLUSTRATIVE EXAMPLE

Some aspects of the plane strain behaviour are now discussed with reference to an example. The value $\nu = 0.25$ is assumed for the elastic Poisson ratio and the following stress states are considered

$$\sigma_x = \bar{\sigma}_0 S \quad \sigma_y = \eta\bar{\sigma}_0 S \quad \tau_{xy} = 0 \tag{48}$$

where $\bar{\sigma}_0$ is the initial yield limit.

A peculiar feature of the plane strain representation is the fictitious hardening contribution. This is present even if the material is perfectly plastic and this case is first examined. The back stress vector coincides with $\boldsymbol{\Pi}$, eqns (30), and, for the value of ν considered, reads

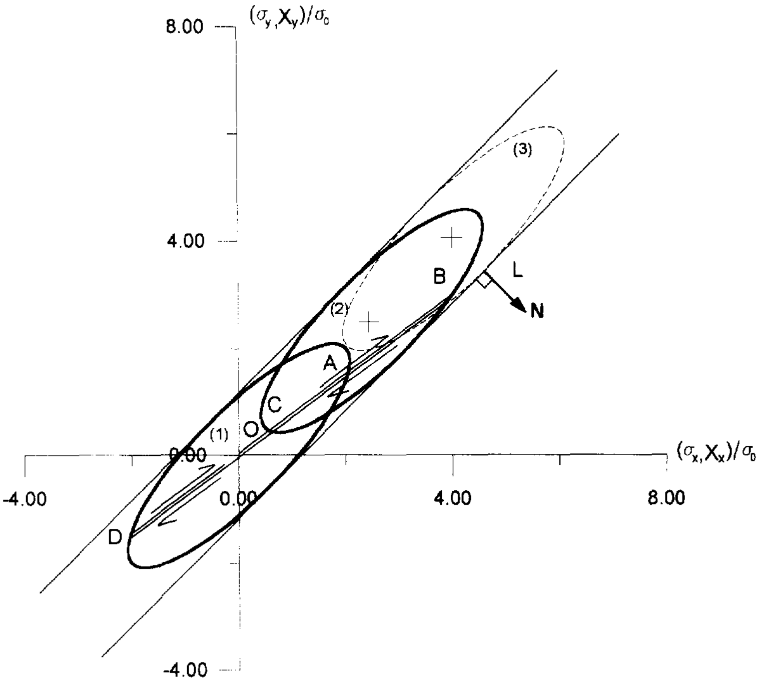


Fig. 1. Evolution of the plane strain yield surface for perfect plasticity. Stress path eqn (48) with $\eta = 0.75$. (1) Initial position; (2) position for $S = 4.0$; (3) limit situation ($S = 8/\sqrt{3}$).

$$\mathbf{X} = \mathbf{\Pi} = 4E(P_x + P_y)\boldsymbol{\mu} = \sigma_0 \pi \boldsymbol{\mu} \qquad \pi = 4\frac{E}{\sigma_0}(P_x + P_y) \tag{49a,b}$$

(the bar over σ_0 was eliminated, the yield limit being constant). Note that the stress path eqns (48) is radial only in the plane, since the transverse stress component σ_z is not proportional to S. In fact, account taken of the above relations, eqn (29) becomes

$$S_z = \frac{\sigma_z}{\sigma_0} = \frac{1+\eta}{4}S + \tfrac{1}{2}\pi. \tag{50}$$

By introducing eqns (48) and (49) in eqn (34a), the following expression for the plane strain yield function is arrived at

$$\Phi = \alpha - \sigma_0 \quad \alpha^2 = \frac{\sigma_0^2}{16}[S^2(13 - 22\eta + 13\eta^2) - 4(1+\eta)S\pi + 4\pi^2]. \tag{51}$$

In the plane σ_x–σ_y, the elastic range is bounded by the instantaneous yield surface $\Phi = 0$, an ellipse centered at the point $X_x = X_y = \pi$, the current value of the back stress parameter eqn (49b). Initially it is $\pi = 0$ and the ellipse is centred at the origin (curve 1 in Fig. 1). When the plastic flow develops, the yield surface translates along the $\sigma_x = \sigma_y$ axis, remaining within the strip

$$-\frac{2}{\sqrt{3}}\sigma_0 \leqslant \sigma_x - \sigma_y \leqslant \frac{2}{\sqrt{3}}\sigma_0 \tag{52}$$

which defines the plane strain limit domain for perfectly plastic materials.

When S increases from zero, the response is elastic up to the value

$$S_e = \frac{4}{\sqrt{13-22\eta+13\eta^2}}. \tag{53}$$

The stress point is now on the boundary of the initial plastic domain (point A in Fig. 1, referring to $\eta = 0.75$, $S_e = 2.049$). Plastic flow may develop, but it is elastically contrasted as long as $\dot{p}_z$ is different from zero and, hence, S may increase; in this process, the stress point keeps in contact with the yield surface and from the condition $\Phi = 0$ the value of π is readily computed. From eqns (51) one obtains

$$\pi = \frac{1+\eta}{2}S - \sqrt{4-3(1-\eta)^2 S^2}. \tag{54}$$

The value, eqn (54), of π is real as long as the term under the square root is positive, i.e. up to the value

$$S_L = \frac{2}{\sqrt{3}}\frac{1}{1-\eta}. \tag{55}$$

The limit situation for the stress path, eqns (48), has now been reached, the values $\sigma_x = S_L\sigma_0$, $\sigma_y = \eta S_L \sigma_0$ being on the boundary of the limit domain eqns (52). For $\eta = 0.75$ it is $S_L = 8/\sqrt{3} = 4.619$ and the in-plane stress components are located at the point L in Fig. 1. The yield surface $\Phi = 0$ is curve 3 (dashed) in the same figure. The two non-vanishing components of vector $\mathbf{N} = \partial\Phi/\partial\boldsymbol{\sigma}$ have now equal values and opposite sign; this implies $\dot{P}_x + \dot{P}_y = \boldsymbol{\mu}^t\dot{\mathbf{P}} = 0$ and, hence, $\dot{p}_z = 0$ [see eqn (28)]: plastic flow involves in-plane components only and collapse is no longer prevented by transverse yielding.

Note that no limit situation exists for $\eta = 1$ ($\sigma_x = \sigma_y = \sigma_0 S$): as eqns (50) and (54) show, in this case it is $\sigma_z = \sigma_0(S-1)$ and with increasing S stresses approach a hydrostatic state. On the other hand, when $\eta = -1$ ($\sigma_x = -\sigma_y$) the values of S_e and S_L coincide and the limit situation is attained as soon as the elastic capabilities are exhausted.

It is of interest to consider loading histories in which S decreases after a value of $\bar{S} > S_e$ has been attained. A complete analysis is now performed for $\eta = 0.75$; the value of the back stress parameter π and of the transverse stress S_z, eqn (50), will be computed.

When S initially increases from zero, the response is elastic up to S_e. Hence

$$0 \leqslant S \leqslant S_e = 2.049: \quad \pi = 0, \quad S_z = \tfrac{7}{16}S. \tag{56a}$$

As the value S_e is exceeded, eqn (54) applies and, for the value of η under consideration, one has

$$S_e \leqslant S \leqslant \bar{S} < S_L: \quad \pi = \tfrac{1}{8}(7S - 2\sqrt{64-3S^2}), \quad S_z = \tfrac{1}{8}(7S - \sqrt{64-3S^2}). \tag{56b}$$

Let the loading phase end at $\bar{S} = 4.0$, when $\bar{\pi} = \pi(\bar{S}) = 2.5$ and $\bar{S}_z = S_z(\bar{S}) = 3.0$. The stress point is now the one indicated as B and the current elastic range is bounded by curve 2 in Fig. 1. When S decreases from $\bar{S}$, the unloading process is elastic, the back stress parameter keeping the value $\bar{\pi}$, until the stress point gets in contact with the yield surface in the reverse direction (point C in Fig. 1). Straightforward computations show that this happens for $S = S_{RY} = 0.590$, and one can write

$$\bar{S} \geqslant S \geqslant S_{RY} = 0.590: \quad \pi = 2.5, \quad S_z = \tfrac{7}{16}S + 1.25. \tag{56c}$$

If S is further decreased, new plastic strains are generated. By proceeding as in the loading case, one obtains

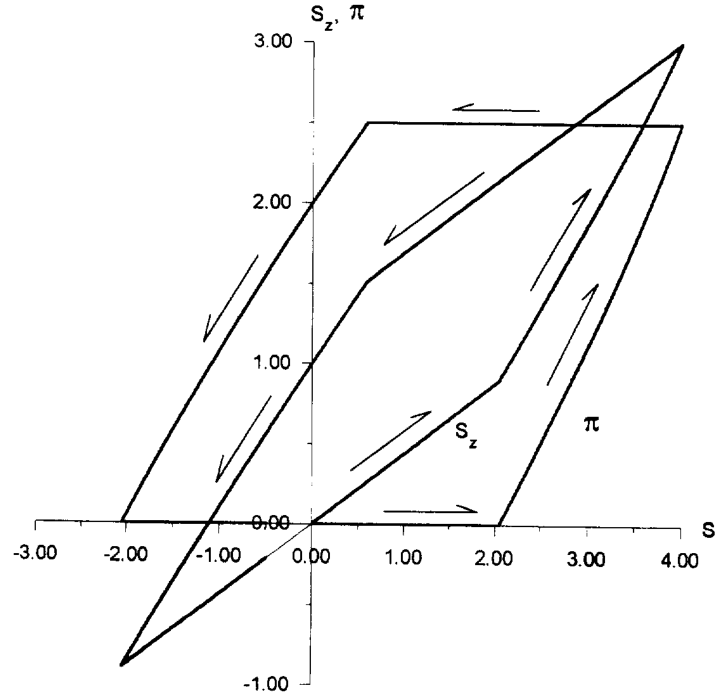


Fig. 2. Back stress (π) and transverse stress (S_z) histories on the cycle *OABCDO* in Fig. 1.

$$S_{RY} \geqslant S \geqslant -S_e = -2.049: \quad \pi = \tfrac{1}{8}(7S+2\sqrt{64-3S^2}), \quad S_z = \tfrac{1}{8}(7S+\sqrt{64-3S^2}). \tag{56d}$$

The unloading process was stopped at $S = -S_e$, when $\pi = 0$ and the yield surface has recovered the original position (point D in Fig. 1). Figure 2 shows how π and S_z evolve along the cycle. These quantities recover the original values when S goes back to zero, but the same is not true for the in-plane plastic strain components. The example is simple enough to permit their closed form computation and the procedure is briefly summarized.

The equivalent in-plane plastic strain measures are first evaluated. In the loading phase, it is $P_x = P_y = 0$ up to $S = S_e$. When this value is exceeded, the normality law eqn (17a) establishes, for the case under consideration

$$\dot{P}_x = \frac{\dot{\lambda}}{32}(6S+\sqrt{64-3S^2}) \qquad \dot{P}_y = \frac{\dot{\lambda}}{32}(-6S+\sqrt{64-3S^2}). \tag{57a,b}$$

By recalling eqn (49a) one obtains on this basis

$$\dot{P}_x+\dot{P}_y = \frac{\dot{\lambda}}{16}(\sqrt{64-3S^2}) = \frac{\sigma_0}{4E}\dot{\pi}$$

and, hence

$$\dot{\lambda} = \frac{4}{\sqrt{64-3S^2}}\frac{\sigma_0}{E}\dot{\pi} = \frac{\sigma_0}{2E}\frac{1}{\sqrt{64-3S^2}}\left(7+\frac{6S}{\sqrt{64-3S^2}}\right)\dot{S} \tag{57c}$$

where $\dot{\pi}$ was computed from the relevant expressions (56b) of π. From eqns (57) two differential equations are obtained, which can be integrated under the initial conditions $P_x(S_e) = P_y(S_e) = 0$, to give

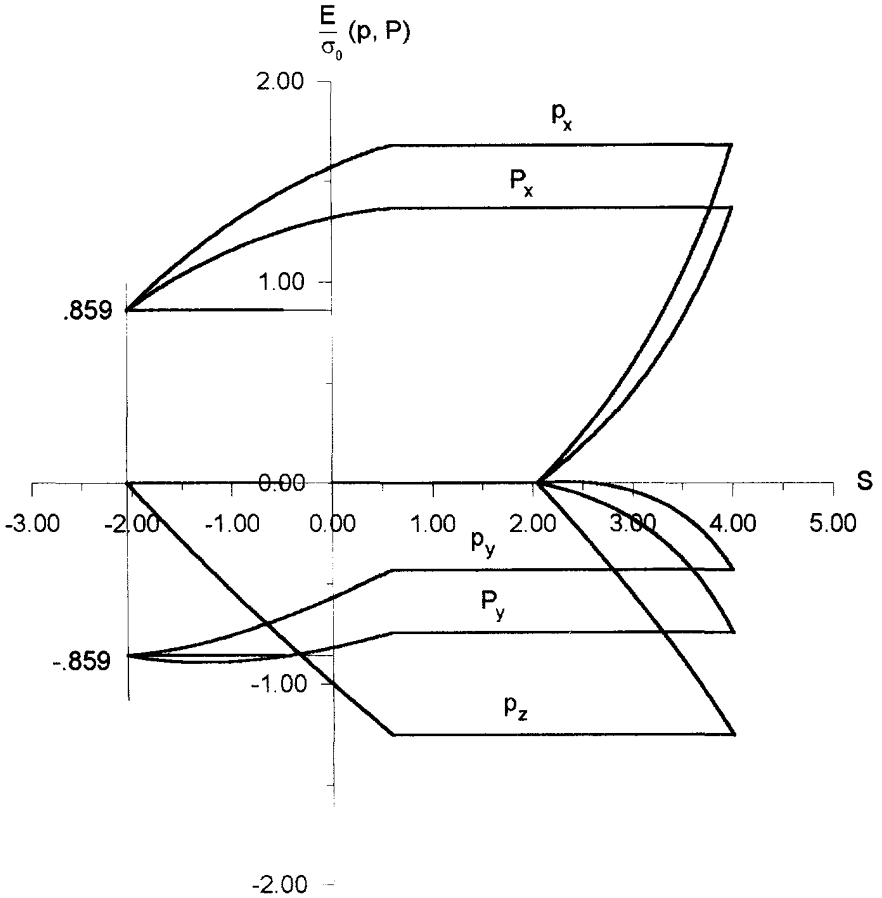


Fig. 3. Equivalent (P_x, P_y) and actual (p_x, p_y, p_z) plastic strain histories on the cycle *OABCDO* in Fig. 1.

$$\frac{E}{\sigma_0}P_x = -\tfrac{5}{64}S-\tfrac{1}{4}\sqrt{64-3S^2}+\frac{\sqrt{3}}{4}\ln\left(\frac{8+\sqrt{3}S}{8-\sqrt{3}S}\right)+1.540$$

$$\frac{E}{\sigma_0}P_y = \tfrac{19}{64}S+\tfrac{3}{16}\sqrt{64-3S^2}-\frac{\sqrt{3}}{4}\ln\left(\frac{8+\sqrt{3}S}{8-\sqrt{3}S}\right)-1.540. \tag{58a}$$

Note that the above values get unbounded for $S \to S_L = 8/\sqrt{3}$. In the unloading process from $\bar{S} = 4.0$, the values $\bar{P}_x = P_x(\bar{S})$, $\bar{P}_y = P_y(\bar{S})$ remain stored for $S \geqslant S_{RY}$. When S further decreases, differential equations similar to the previous ones can be produced and integrated under the conditions $P_x(S_{RY}) = \bar{P}_x$, $P_y(S_{RY}) = \bar{P}_y$. One obtains

$$\frac{E}{\sigma_0}P_x = -\tfrac{5}{64}S+\tfrac{1}{4}\sqrt{64-3S^2}+\frac{\sqrt{3}}{4}\ln\left(\frac{8+\sqrt{3}S}{8-\sqrt{3}S}\right)-0.681$$

$$\frac{E}{\sigma_0}P_y = \tfrac{19}{64}S-\tfrac{3}{16}\sqrt{64-3S^2}-\frac{\sqrt{3}}{4}\ln\left(\frac{8+\sqrt{3}S}{8-\sqrt{3}S}\right)+0.681. \tag{58b}$$

The actual plastic strain values are computed from eqns (26) and (28), which for $\nu = 0.25$ read

$$p_x = P_x-\tfrac{1}{4}p_z \quad p_y = P_y-\tfrac{1}{4}p_z \quad p_z = -2(P_x+P_y).$$

Figure 3 illustrates the evolution of equivalent and actual plastic strains for the example considered. Note that for $S = -S_e$ it is $P_x = p_x = .859(\sigma_0/E)$ and $P_y = p_y = -0.859(\sigma_0/E)$. Only p_z goes back to zero after the cycle.

The plane strain picture can be interpreted as follows. In the usual, three-dimensional representation, the yield function is an unmoving cylinder in the principal stress space [its projection on the plane $\sigma_z = 0$ is the limit domain eqn (52)]. Initially, the stress point moves on the plane $\sigma_z = \nu(\sigma_x+\sigma_y)$ (along the line $S_z = 7S/16$ for the values of ν and η in the example); the intersection of this plane with the von Mises cylinder, projected on $\sigma_z = 0$, produces the initial plane strain yield surface. When the stress point gets in contact with the cylinder, plastic strains develop and eqn (12b) establishes $\sigma_z = \nu(\sigma_x+\sigma_y)-Ep_z$ ($S_z = 7S/16+\pi/2$ in the example), showing that the stress plane translates parallel to itself; as a consequence, its intersection with the cylinder, when projected on the plane, undergoes a motion which appears as a directional, kinematic hardening.

When material hardening is present, its effects combine with the fictitious plane strain contribution. Some typical behaviours are illustrated in Fig. 4, referring to $\eta = 0.75$ and S increasing from zero to 4.619, the limit value for perfect plasticity. The initial elastic limit is not affected by hardening and corresponds to the dashed curves labelled as 1. For comparison, the perfectly plastic response is again illustrated in Fig. 4(a); at the final value of S, the yield surface is curve 2.

Figure 4(b) refers to linear kinematic hardening, with $c = 0.1E$ assumed in eqn (35). For $S = 4.619$, the yield surface is now curve 3: the instantaneous elastic range still undergoes a rigid translation, which combines the fictitious, directional plane strain contribution (significantly contrasted by material hardening) with an actual kinematic one. Isotropic hardening was next included by assuming $\sigma_0(\kappa) = \bar{\sigma}_0(1+\eta\kappa)$, with $\eta = 0.1$ and the strain hardening definition for κ. When only this contribution is present, the evolution of the yield surface superimposes the isotropic hardening expansion to the directional translation (curve 4 in Fig. 4(c) for $S = 4.619$). Curve 5 in Fig. 1(d) shows the yield surface for the same value of S when both kinematic and isotropic hardening are considered.

7. COMMENTS AND CONCLUSIONS

The elastic–plastic rate law associated to the plane strain form eqns (34), of the von Mises yield function is completely defined by the expressions eqns (38) of vector **N** and eqns (43) of hardening parameter H. It could be verified that these relations are just a particular case of the general expressions (19) and, in fact, they could also have been obtained by specializing the latter to the case presently considered. Their direct derivation, however, is somewhat simpler and, it is felt, permits a better understanding of the meaning of operations and results.

The plane strain law maintains the essential features of the original one. In particular Drucker's postulate conditions are complied with if obeyed by the material law. That is

$$(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})^{\mathrm{T}}\dot{\mathbf{P}} \geqslant 0 \qquad \forall\hat{\boldsymbol{\sigma}} \text{ such that } \Phi(\hat{\boldsymbol{\sigma}}, \mathbf{P}, \boldsymbol{\kappa}) \leqslant 0 \tag{59a}$$

if

$$(\sigma_{ij}-\hat{\sigma}_{ij})\dot{\varepsilon}^p_{ij} \geqslant 0 \quad \forall\hat{\sigma}_{ij} \text{ such that } \Phi(\hat{\sigma}_{ij}, \varepsilon^p_{ij}, \kappa) \leqslant 0 \tag{59b}$$

and

$$\dot{\boldsymbol{\sigma}}^{\mathrm{T}}\dot{\mathbf{P}} \geqslant 0 \quad \forall\dot{\mathbf{P}} \tag{60a}$$

if

$$\dot{\sigma}_{ij}\dot{\varepsilon}^p_{ij} \geqslant 0 \quad \forall\dot{\varepsilon}^p_{ij}. \tag{60b}$$

The above result was established in Corradi and Gioda (1979) for the general plane strain expressions eqn (16)–(19) and no further proof is needed. Its validity in the present context, however, is easily checked. Equation (59a) implies convexity for the instantaneous elastic domain $\Phi \leqslant 0$ and outward normality for the equivalent, in-plane plastic strain rates $\dot{\mathbf{P}}$.

The first condition is ensured since matrix **M**, as defined by eqn (34b), is positive definite for all allowable values of elastic Poisson ratio, the second is established by eqns (17a), (18) and (38). On the other hand, eqn (50a) states that the plane strain hardening parameter H is non-negative, which is ensured by eqn (43), expressing H as the sum of two material hardening coefficients $h = H_I + H_K$, non-negative if eqn (50b) holds, and of the fictitious plane strain contribution H_E, also non-negative, as shown in eqn (45).

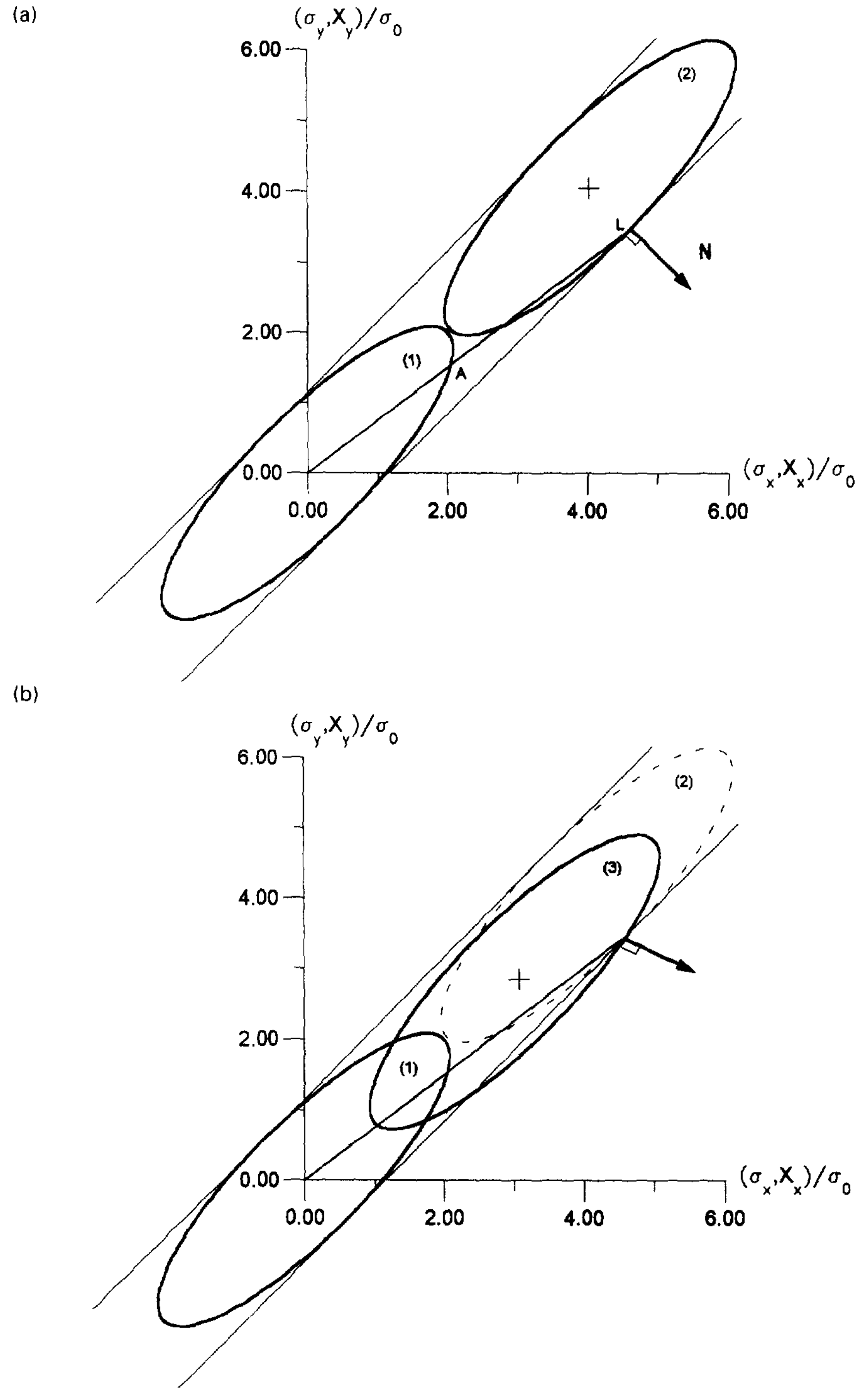
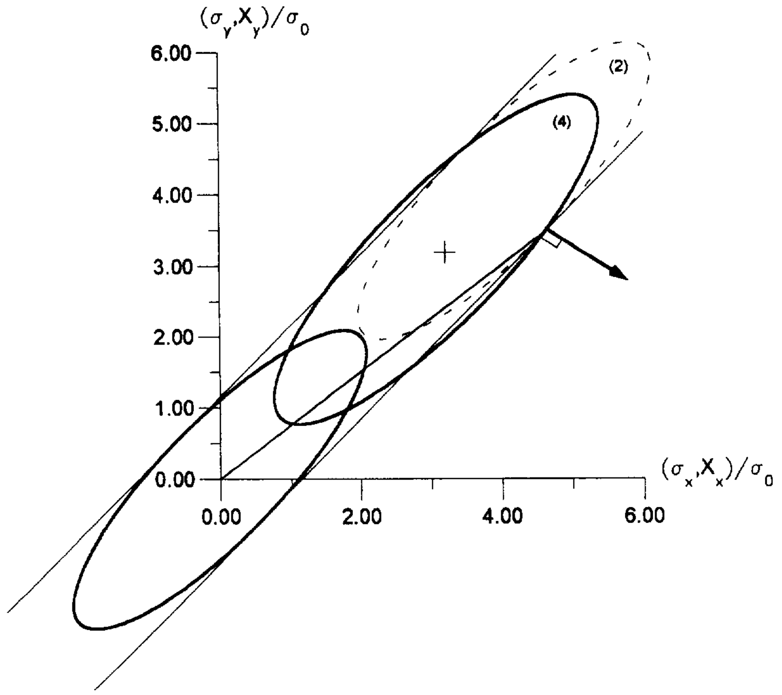


Fig. 4. Initial and final yield surfaces for a radial path from $S = 0$ to $S = 8/\sqrt{3}$ ($\eta = 0.75$). (a) Perfect plasticity; (b) kinematic hardening; (c) isotropic hardening; (d) combined hardening. + = final position of the back-stress vector.

(c)



(d)

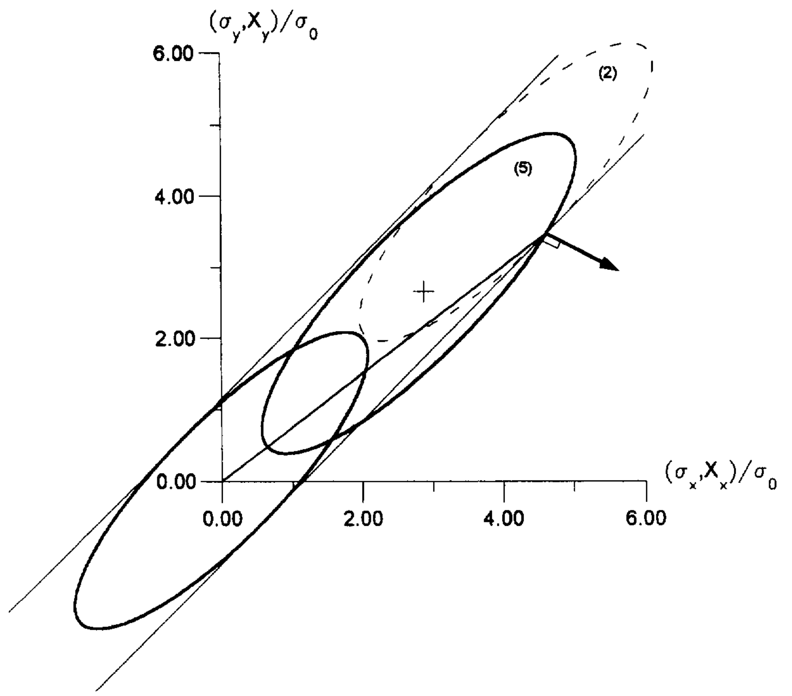


Fig. 4—*continued.*

In this paper material softening ($h < 0$) was not considered, but it could be easily incorporated. It is worth noting that in any case, the additional plane strain term H_E has a stabilizing effect.

With the exception of the definition of matrices, and the meaning of the variables involved, the plane strain law is identical to a plane stress one and can be handled in the same way. Together with the plane form of the equilibrium and geometric compatibility conditions, it permits the formulation of the plane strain elastic–plastic problem in terms of in-plane variables only. As for the elastic case, the problem differs from the equivalent, plane stress one only in the expression of constitutive parameters. The information on plastic strains is produced in terms of equivalent in-plane values **P**; they are in a sense, artificial definitions, but actual plastic strains can easily be obtained *a posteriori* through eqns (26) and (28).

Finally, it must be mentioned that the procedure could also be applied to other yield conditions, such as Tresca's, which imply that plastic flow does not entail volume changes. The resulting expressions, however, are not equally simple and the advantages connected with their use are questionable.

Acknowledgement—The financial support of the Italian Ministry of University and Research (MURST) is gratefully acknowledged.

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